

Beyond the rotating wave approximation: conditional squeezing and quantum non-demolition readout in the Jaynes-Cummings model

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Abstract: A first-principles Magnus expansion treatment of the Jaynes-Cummings model beyond the Rotating Wave Approximation reveals second-order energy shifts and $\hat{\sigma}_z$ -dependent field squeezing, suggesting a natural route towards a quantum non-demolition readout.

1. Introduction

Jaynes-Cummings Model provides a foundational description of the interaction between a two-level atom and a single-mode cavity field. In most practical treatments, the Rotating Wave Approximation (RWA) is employed to simplify the dynamics by neglecting the contributions of rapidly oscillating counter-rotating terms in the Hamiltonian. However, less attention has been paid to the role of counter-rotating terms in generating non-classical squeezed light within the simplest two-level atom framework. Squeezed states of light, which was first theorized by Caves [1], has proven applications in gravitational wave detectors [2]. This work derives such a squeezing phenomenon as beyond the RWA phenomenon from the first-principle without any phenomenological assumptions. Moreover, Quantum Non-Demolition (QND) measurements have been a topic of particular interest in quantum information since such measurements keep the state of the qubit unaltered while making a measurement [3]. Our work shows how the squeezing term yielded by the second-order Magnus Expansion provides us with a path towards a QND measurement device.

2. The Magnus Expansion on the Jaynes-Cummings Hamiltonian

The Magnus Expansion describes the time-evolution operator of the Schrodinger Equation as an exponential involving a series of nested commutators as follows.

$$\hat{U}(t) = \exp(\hat{\Omega}_1(t) + \hat{\Omega}_2(t) + \dots) \quad (1)$$

$$\hat{\Omega}_1(t) = -\frac{i}{\hbar} \int_0^t \hat{H}(t_1) dt_1 \quad (2)$$

$$\hat{\Omega}_2(t) = \frac{1}{2} \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 [\hat{H}(t_1), \hat{H}(t_2)] \quad (3)$$

This presentation will focus on the results obtained at the second order in the Magnus expansion, revealing $\hat{\sigma}_z$ -dependent field squeezing and the energy shifts. Motivated by these findings, we are extending the analysis to third- and fourth-order terms, which are anticipated to generate non-Gaussian corrections and will be briefly addressed if completed.

The Jaynes-Cummings Hamiltonian governing the interaction of a two-level system and a single-mode cavity electromagnetic field in the rotated frame is as follows.

$$\hat{H}(t) = \hbar g (i\hat{a}^\dagger \hat{\sigma}_- e^{i\Delta t} - i\hat{a} \hat{\sigma}_+ e^{-i\Delta t} + i\hat{a}^\dagger \hat{\sigma}_+ e^{i\Sigma t} - i\hat{a} \hat{\sigma}_- e^{-i\Sigma t}) \quad (4)$$

In equation 4, \hbar represents the reduced Planck's constant, g the strength of coupling between the field and the atom, \hat{a}^\dagger and \hat{a} the photon creation and annihilation operators of the field, $\hat{\sigma}_+$ and $\hat{\sigma}_-$ the raising and lowering operators of the two-level system, and $\Delta = \omega - \omega_0, \Sigma = \omega + \omega_0$, where ω represents the field frequency and ω_0 the transition frequency of the two-level atom.

3. Results

Applying equation 3 to the Jaynes-Cummings Hamiltonian, we present the full second-order time evolution operator as follows.

$$\hat{\Omega}_2(t) = ig^2(\hat{a}^\dagger \hat{a} + \frac{1}{2})\left(\frac{t}{\Delta} - \frac{t}{\Sigma} + \frac{\sin(\Sigma t)}{\Sigma^2} - \frac{\sin(\Delta t)}{\Delta^2}\right)\hat{\sigma}_z + \frac{1}{2}(\zeta^*(t)\hat{a}^2 - \zeta(t)(\hat{a}^\dagger)^2)\hat{\sigma}_z \quad (5)$$

In equation 5, $\hat{\sigma}_z$ operator represents the Pauli-z operator and $\zeta(t)$ the time-dependent squeezing coefficient with the full form shown in equation 6. The first term in the sum represents time-dependent frequency shifts, including the AC-Stark shift governed by Δ and $\hat{a}^\dagger \hat{a}$, the second term the Bloch-Siegert shift governed by Σ and $\hat{a}^\dagger \hat{a}$, and the vacuum-induced energy shift governed by the $\frac{1}{2}$ term. The second term represents the squeezing operator formally defined as $\hat{S}(t) = \exp(\frac{1}{2}(\zeta^*(t)\hat{a}^2 - \zeta(t)(\hat{a}^\dagger)^2))$ [4]. Since the squeezing term depends on $\hat{\sigma}_z$ operator, the squeezing coefficient will have a different sign depending on the state of the atom. In other words, our results suggest that one can infer the state of the atom by measuring the phase of the outgoing squeezed light. Such a prediction opens possibilities of a QND detector. Moreover, the closed-form expression of the squeezing coefficient is as follows.

$$\zeta(t) = g^2 \frac{\omega_0 e^{2i\omega t} - \omega e^{i\Sigma t} + \omega e^{i\Delta t} - \omega_0}{\omega(\omega^2 - \omega_0^2)} \quad (6)$$

Moreover, the operators $\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \frac{1}{2})$, $\hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2$, and $\hat{K}_- = \frac{1}{2}\hat{a}^2$ generate a representation of the SU(1,1) Lie algebra. As a result, the exponentiation of the second-order Magnus operator admits a closed-form unitary evolution [5].

4. Conclusion

This work provides a first-principles Magnus Expansion treatment of the Jaynes-Cummings model beyond the RWA ¹. The work identifies an explicit two-photon term responsible for the conditional quadrature squeezing of the field. While squeezing of a two-level system is explicitly studied, the present approach yields a compact analytical expression for the squeezing coefficient up to the second order without any phenomenological approximations. The dependence of the squeezing phase on the atomic state suggests a natural route towards a quantum non-demolition readout via homodyne detection. More effectively, this work highlights how beyond the RWA analyses encode measurement devices even in minimal light-matter models.

References

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